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# Rigorous Model of Classical Spacetime Foam

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If in the gravity quantization process one changes from the smooth manifold category to a more general category, qualitatively new features can appear. To illustrate this, we construct a geometrically precise but physically naive model of a classical "spacetime foam" and discuss the consequences of the principle of general covariance and the equivalence principle in this more general setting. We also show how Einstein's equations can be defined on this "spacetime foam."

## INTRODUCTION

At the beginning of our century physicists were confronted with two critical situations: the crisis of the "electrodynamics of moving bodies" and the crisis of black body radiation. Before giving his ingenious solution to the problem of the electrodynamics of moving bodies, Einstein formulated operational definitions of those concepts which had been involved in this problem, and it was precisely these definitions that showed him the correct solution. Unfortunately, since the energies in which the unification of quantum and gravity physics is supposed to occur transcend the actual possibilities by many orders of magnitude, we cannot hope for any truly operational definitions in this field. Many authors look for "theoretical observables" and their corresponding "theoretically operational definitions" (see the discussion in Rovelli, 1991a, b). This line of research seems to be interesting, but so far it has not brought any spectacular results. In these circumstances, the doctrine of a correspondence with classical physics, elaborated by Bohr in his struggle to resolve the second crisis, was both a guiding idea and a practical tool to formulate quantum laws. Of course, it is now a common practice that the newly proposed theory must, through a kind of limiting

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process, have a correspondence with former theories, but our suggestion is that we should take Bohr's strategy more seriously and thoroughly discuss—in the light of the correspondence idea—physical principles upon which the present theories are based and which are believed to play a role in the quantum gravity regime. It is a kind of dogma among physicists that the future theory of quantum gravity should be *generally covariant*, and only very few doubt (for instance, Taylor, 1979; Candelas and Sciama, 1984) that it should deviate from Einstein's principle of equivalence. In the present paper we present a naive physical model of a classical "spacetime foam" which, however, is based on a rigorous mathematical structure, providing a good conceptual framework to critically discuss these two principles. Models like the present one could be regarded as paving the way for more realistic quantum models. The idea of a quantum spacetime foam was proposed by Hawking (1982) and subsequently resulted in the path integration approach to the quantization of gravity (Hartle and Hawking, 1983). The more radical idea of a "foam of topologies" as a phase space for quantum gravity was presented by Isham (1989). In contrast to these proposals (and to some others as well), we do not regard our classical "spacetime foam" model as even a tentative approach to reality, but rather as a mathematically precise context allowing us to discuss some interconnections between geometry and its physical interpretation.

The geometric content of the general covariance principle (GC), when looked upon from a general enough geometric perspective, may be expressed in the following way. Let $\mathscr{E}$ be the set of all solutions of Einstein's field equations, and *Diff* the group of all spacetime diffeomorphisms. GC asserts that the physical meaning should be ascribed to $\hat{\mathscr{E}} := \mathscr{E}/Diff$ rather than to $\mathscr{E}$. Let further $\mathscr{G}$ be a category of some geometric spaces (possibly more general than the category of smooth manifolds). Any morphism $\eta \in \mathscr{G}$ such that there exists the inverse morphism $\eta^{-1} \in \mathscr{G}$ is called an *isomorphism* in the category $\mathscr{G}$. Let us assume that Einstein's equations have been generalized in such a way that some objects of the category $\mathscr{G}$ are their solutions. Let $\mathfrak{X}$ be the space of all such solutions and *Iso* the group of all isomorphisms in the category $\mathscr{G}$. In this context GC would assert that the physical meaning should be ascribed to $\hat{\mathfrak{X}} := \mathfrak{X}/Iso$ rather than to $\mathfrak{X}$. Of course, if $\mathscr{G}$ is the category of smooth manifolds, then $Iso = Diff$. Our classical "spacetime foam" model is covariant in this sense.

In Section 1, the *modeling category* is introduced in terms of which (generalized) charts may be defined on a very general type of spaces called *sheaf spaces* (Berezin 1983). Smooth manifolds are spaces modeled on $\mathbf{R}^n$; in this case, all our definitions go "smoothly" to the usual ones.

In Section 2, we construct our toy model of a classical spacetime. The model describes a foam similar to the foam which is formed when soap powder is dissolved in water: two-dimensional smooth sectors are joined together along singular one-dimensional edges and zero-dimensional vertices to form a three-dimensional configuration. To such a foam we add one more (temporal) dimension. It is assumed that test particles follow classical trajectories on the background of the above-described spacetime. The modeling category in this case is the space consisting of four intersecting 3-spaces naturally spanned by the positive Minkowski coordinate half-axes. We develop differential geometry on such spaces and construct the group *Iso* for them (in Section 3) which allows us to study the consequences of GC in this particular case.

The equivalence principle (EP) was thoroughly analyzed by many authors (e.g., Weinberg, 1972; Torretti, 1983; Raine and Heller, 1981). EP in its *weak form* (called also the principle *of the universality of free fall*) asserts that "in the absence of non-gravitational influences, test bodies released from the same point at the same time with the same initial velocity will follow identical trajectories in spacetime independently of their composition and internal structure" (Raine, 1981, p. 94). EP in its *strong form* says that "in a freely falling frame the laws of non-gravitational physics assume the standard form they have in the absence of gravity" (Raine, 1981, p. 94), i.e., that the gravitational field can always locally be transformed away. In Section 4, we demonstrate that in our toy model of the classical "spacetime foam" both versions of EP break down on singular edges and vertices. However, the model itself suggests how EP should be generalized.

In Section 5, we show that Einstein's field equations can consistently be defined on the classical "spacetime foam" or, in other words, that the Lorentz "spacetime foam" can be a solution of suitably generalized Einstein equations. In Section 6, we analyze the fact that a test particle, on approaching an edge or a vertex in the "spacetime foam," has no warning that it is approaching the singularity. In this sense, the singular part of the spacetime curvature which is concentrated in singularities gives rise to a short-range (strictly localized) force.

We think that, besides clarifying certain physical concepts, the present work also could be interesting from the purely geometric point of view, especially Section 1, in which some standard concepts are essentially generalized.

## 1. SHEAF SPACES AND MODELING CATEGORIES

By a *sheaf space* we understand a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ a sheaf of algebras on $X$. $\mathcal{O}_X$ is called a *structural sheaf*. The *morphism* from a sheaf space $(X, \mathcal{O}_X)$ to a sheaf space $(Y, \mathcal{O}_Y)$ is the pair of mappings

$$(f, \phi): \quad (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $f: X \to Y$ is a continuous mapping, and $\phi: f^*\mathcal{O}_Y \to \mathcal{O}_X$ is a mapping of sheaves of algebras.

Let **K** denote **R** or **C**. In the following, we shall consider only **K**-algebras (see, for instance, Berezin, 1983). A sheaf space $(X, \mathcal{O}_X)$ is said to be a **K**-*sheaf space* if $\mathcal{O}_X$ is a sheaf of **K**-algebras on $X$. By $\mathcal{O}_{X,x}$ we denote the stalk of $\mathcal{O}_X$ at $x \in X$. A morphism from a sheaf space $(X, \mathcal{O}_X)$ to a sheaf space $(Y, \mathcal{O}_Y)$ is said to be a *morphism of* **K**-*sheaf spaces* if, for every $x \in X$, the ring homomorphism

$$\phi_x: \quad \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a homomorphism of **K**-algebras. **K**-sheaf spaces as objects together with morphisms of **K**-sheaf spaces as morphisms form the *category of* **K**-*sheaf spaces*.

In the following, we shall consider only *local* **K**-sheaf spaces, i.e., **K**-sheaf spaces $(X, \mathcal{O}_X)$ such that in every $\mathcal{O}_{X,x}$ (for every $x \in X$) there exists the unique maximal ideal.

Now, we are ready to define the modeling category and spaces of a given type which are modeled in terms of this category.

*Definition 1.1.* A category $\mathcal{M}$ of **K**-sheaf spaces, such that if a **K**-sheaf space $M$ is an object of $\mathcal{M}$, then any of subspaces of $M$ is also an object of $\mathcal{M}$, is a *modeling category* (called also *category of modeling spaces*). Its objects and morphisms are called *modeling objects* (or *modeling spaces*) and *modeling morphisms*, respectively.

*Definition 1.2.* Let $\mathcal{M}$ be a fixed modeling category. The class of **K**-sheaf spaces $(X, \mathcal{O}_X)$, together with their morphisms, is a *category of* **K**-*sheaf spaces of the type* $\mathcal{M}$, if the following conditions are satisfied:

(i) For every $x \in X$ there exists an open neighborhood $U$ of $x$ such that there is an isomorphism onto its image of **K**-sheaf spaces

$$(f, \phi): \quad (U, \mathcal{O}_X|U) \to (L, \mathcal{O}_L)$$

where $(L, \mathcal{O}_L)$ is a modeling space, $f: U \to L$ is continuous mapping, and $\phi: f^*\mathcal{O}_L \to \mathcal{O}_X|U$. We call $(f, \phi)$ an $\mathcal{M}$-*chart* on $X$.

(ii) If there is another $\mathcal{M}$-chart on $X$

$$(g, \psi): \quad (V, \mathcal{O}_X|V) \to (K, \mathcal{O}_K)$$

where $(K, \mathcal{O}_K)$ is another modeling space, such that $U \cap V \neq \emptyset$, then the morphism $(h, \chi)$ determined by the commutative diagram

$$\begin{array}{ccc} & (U \cap V, \mathcal{O}_X|U \cap V) & \\ {\scriptstyle (f', \phi')} \swarrow & & \searrow {\scriptstyle (g', \psi')} \\ (L', \mathcal{O}_L|L') & \xrightarrow{(h, \chi)} & (K', \mathcal{O}_K|K') \end{array}$$

where $L' = f(U \cap V)$, $K' = g(U \cap V)$, is postulated to be a modeling morphism. The mappings $(f', \phi')$ and $(g', \psi')$ are restrictions of the $\mathcal{M}$-charts $(f, \phi)$ and $(g, \psi)$ to $U \cap V$. The morphism $(h, \chi)$ (which in fact is an isomorphism) is called the *transition morphism* of the $\mathcal{M}$-charts $(f, \phi)$ and $(g, \psi)$ on $X$.

The set $\mathcal{A}$ of all $\mathcal{M}$-charts on $X$ is called the *$\mathcal{M}$-atlas* on the **K**-sheaf space $(X, \mathcal{O}_X)$.

*Definition 1.3.* The set $\mathcal{A}$ of all $\mathcal{M}$-charts on $X$ is a *maximal $\mathcal{M}$-atlas* on the **K**-sheaf space $(X, \mathcal{O}_X)$. The subset $\mathcal{A}_0$ of the maximal $\mathcal{M}$-atlas $\mathcal{A}$ on $(X, \mathcal{O}_X)$, such that the domains of $\mathcal{M}$-charts belonging to $\mathcal{A}$ cover $X$, is an $\mathcal{M}$-atlas on $(X, \mathcal{O}_X)$.

We also need the concept of representations of mappings of **K**-sheaf spaces in different $\mathcal{M}$-charts. Let

$$(F, \Phi): \quad (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of **K**-sheaf spaces of the type $\mathcal{M}$, and let $(f, \phi)$ and $(g, \psi)$ be $\mathcal{N}$-charts on $U \subset X$ and $V \subset Y$, respectively. We say that the morphism $(a, \alpha)$ is a *representation* of $(F, \Phi)$ in the $\mathcal{M}$-charts $(f, \phi)$ and $(g, \psi)$ if $(a, \alpha)$ is determined by the following commutative diagram:

$$\begin{array}{ccc} (U, \mathcal{O}_X|U) & \xrightarrow{(F, \Phi)|U} & (V, \mathcal{O}_Y|V) \\ (f, \phi)\big\downarrow & & \big\downarrow(g, \psi) \\ (L, \mathcal{O}_L) & \xrightarrow[(a, \alpha)]{} & (K, \mathcal{O}_K) \end{array}$$

If $F(U) \not\subset V$, one must restrict the $\mathcal{M}$-chart $(f, \phi)$ to $U \cap F^{-1}(V)$. If $(a, \alpha)$, for any $\mathcal{M}$-chart, is a morphism in the category $\mathcal{M}$, then $(F, \Phi)$ is said to be a *morphism of the type $\mathcal{M}$*.

*Definition 1.4.* Any morphism of the type $\mathcal{M}$

$$(F, \Phi): \quad (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that there exists the inverse morphism to $(F, \Phi)$ of the type $\mathcal{M}$, is called an *isomorphism of the type $\mathcal{M}$*.

The set of all isomorphisms of the type $\mathcal{M}$ of $(X, \mathcal{O}_X)$ into itself will be denoted by $Iso(X, \mathcal{O}_X)$ or simply by $Iso$ if there is no danger of misunderstanding. It can be easily seen that $Iso(X, \mathcal{O}_X)$ with the natural operation of the morphism composition and the neutral element $(id_X, id_{\mathcal{O}_X})$ forms a group.

It can be easily seen that any subset $A \subset X$ of a **K**-sheaf space $(X, \mathcal{O}_X)$ itself becomes, in a natural way, a **K**-sheaf space if one considers the sheaf $\mathcal{O}_X | A$ on $A$. This allows us to introduce the following definition of modeling spaces, which turns out to be more easily applicable in many cases.

*Definition 1.5.* The category of all open subsets of the **K**-sheaf space $(L, \mathcal{O}_L)$ with inclusions as morphisms is called the *strictly modeling category.*

The last property means that if $U \subset V \subset L$, and $U$ and $V$ are open in $L$, then the set of morphisms from $U$ to $V$ is a one-element set consisting of the inclusion $\iota_U$; if $U \not\subset V$, the set of morphisms from $U$ to $V$ is empty. Obviously, open subsets with suitable restricted sheaves of algebras are open subspaces of the **K**-sheaf space $(L, \mathcal{O}_L)$. It can be easily seen that any strictly modeling category has the following property: any open subset of any object of this category is an object of this category. Evidently, the strictly modeling category for $n$-dimensional smooth manifolds is the category of all open subsets of the **R**-sheaf spaces $(\mathbf{R}^n, \mathscr{E}_n)$, as discussed in the following example.

*Example 1.1.* *Smooth manifolds.* Let us consider the **R**-sheaf space $(\mathbf{R}^n, \mathscr{E}_n)$, where $\mathscr{E}_n$ is the sheaf of algebras such that $\mathscr{E}_n(U)$ is the algebra of all smooth, real-valued functions on an open subset $U$ of $\mathbf{R}^n$. We define the category $\mathcal{M}_0$: its objects are subspaces of the **R**-sheaf spaces $(\mathbf{R}^n, \mathscr{E}_n)$, $n = 0, 1, 2, \ldots$, and its morphisms are the pairs of mappings

$$(f, \phi): \quad (U, \mathscr{E}_n | U) \to (V, \mathscr{E}_m | V)$$

where $U$ and $V$ are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, and $f: U \to V$ is of class $C^\infty$ in the usual sense. It can be easily seen that the category of spaces of the type $\mathcal{M}_0$ is the category of smooth manifolds. The group of isomorphisms in this category coincides with the usual group of diffeomorphisms.

*Example 1.2.* *Supermanifolds* (see Berezin, 1983). Let $m, n > 0$ be fixed integers, and let $\Lambda^m$ be the Grassmann algebra over the field **K** (= **R** or **C**) with the generators $\xi_1, \ldots, \xi_m$. We define the sheaf $\mathcal{S}^{n,m}$ on $\mathbf{R}^n$, $U \mapsto \mathcal{S}^{n,m}(U)$, where $U \subset \mathbf{R}^n$ and $\mathcal{S}^{n,m}(U)$ is the **R**-algebra of all $C^\infty$-functions on $U$ with values in $\Lambda^m$. It can be shown that $\mathcal{S}^{n,m}(U)$ are $Z_2$-graded algebras.

We define the modeling category $\mathcal{S}$: its objects are open subspaces of the **R**-sheaf spaces $(\mathbf{R}^n, \mathcal{S}^{n,m})$, $p, q = 0, 1, 2, \ldots$, and its morphisms are any mappings $(f, \phi)$ of these **R**-sheaf spaces which preserve $Z_2$-gradation. Any **R**-sheaf space of the type $\mathcal{S}$ is called a *supermanifold.* Isomorphisms in the category of supermanifolds are any **R**-isomorphisms between supermanifolds that preserve $Z_2$-gradation. [We should notice that DeWitt (1984) and Choquet-Bruhat (1989) give slightly different definitions of supermanifolds. However, these definitions could also be translated into the language of modeling spaces.]

## 2. CLASSICAL SPACE-TIME FOAM

The goal of this section is to construct a strict mathematical model of a classical spacetime foam which would be a special case of the conceptual scheme discussed in the preceding section.

First, we define a strictly modeling category $\mathcal{M}_0$. It is given by the subspace

$L =$

$$\{(t, x, y, z) \in \mathbf{R}^4\text{:} \quad (t = 0 \vee x = 0 \vee y = 0 \vee z = 0) \wedge t, x, y, z \geq 0\} \subset \mathbf{R}^4$$

and all its open subsets. If $\eta$ is the Minkowski metric on $\mathbf{R}^4$, then $\eta_L = \iota^*{}_L\eta$ is the Lorentz metric on $L$. One can represent $L$ as the sum $L = L_0 \cup L_1 \cup L_2 \cup L_3$, where $L_\alpha$, $\alpha = 0, 1, 2, 3$, are 3-dimensional spaces naturally spanned by the positive half-axes of the Minkowski coordinate system. Of course, the Lorentz metric can be pulled back to any of them: $\eta_{L_\alpha} = \iota^*{}_{L_\alpha}\eta_L$. We also have, for instance, $\eta_{L_1 \cap L_2} = -dt^2 + dz^2$, $\eta_{L_0 \cap L_1 \cap L_2} = dz^2$, etc., and $\eta_{L_0 \cap L_1 \cap L_1 \cap L_3} = \{0\}$ with the obvious notation.

*Definition 2.1.* A *(classical) spacetime foam* is a paracompact and Hausdorff topological space $(M, \tau)$ with an $\mathcal{M}_0$-atlas $AtlM$ the $\mathcal{M}$-charts of which are local homeomorphisms onto open sets of the modeling space $L$ such that the transition from one $\mathcal{M}$-chart of the atlas $AtlM$ to another $\mathcal{M}$-chart of this atlas is an isomorphism of the category $\mathcal{M}_0$.

Of course, isomorphisms of the category $\mathcal{M}_0$ are elements of the Poincaré group acting on $L$ and preserving the categorical structure of $L$ (see Section 3 below).

The topological space $(M, \tau)$ can easily be changed into a sheaf space. We define the structural sheaf in the following way: For any $U \in \tau$ let

$$\mathscr{C}(U) = \{f\colon U \to \mathbf{R}\text{: } \forall_{\phi \in AtlM} f \circ \phi^{-1} \in C^\infty(\phi(U \cap D_\phi))\}$$

where $\phi$ has the domain $D_\phi$, and the structural sheaf on $(M, \tau)$ is given by $\mathscr{C}\colon U \mapsto \mathscr{C}(U)$. Thus we have a sheaf space $(M, \mathscr{C})$. [From now on the spacetime foam will also be denoted by $(M, \tau, \mathscr{C})$.]

It can be easily seen that for any $U \in \tau$, $n \in \mathbf{N}$, $f_1, \ldots, f_n \in \mathscr{C}(U)$, and $\omega \in C^\infty(\mathbf{R}^n)$, the composition $\omega \circ (f_1, \ldots, f_n)$ is an element of $\mathscr{C}(U)$. Therefore, the sheaf space $(M, \mathscr{C})$ is also a structured space (Heller and Sasin 1994, 1995), and we can use all differential geometric tools developed for structured spaces to study the structure of our spacetime foam. However, to make the present paper self-contained, in the following we shall introduce all necessary theoretical concepts as especially adapted to the present situation.

By *singular points* (or *singularities*) we shall understand all those points of the spacetime foam $(M, \tau)$ at which its manifold structure breaks down. Let $SingM$ be the set of all singular points of the spacetime foam and $RegM$ the set of all its regular (nonsingular) points. It can be easily seen that $SingM$ can be divided into the following types:

$p \in SingM$ is of the type $K_{0,1,2,3}$ if there is an $\mathcal{M}$-chart $\phi \in AtlM$, $p \in D_\phi$, such that $\phi(p) \in L_0 \cap L_1 \cap L_2 \cap L_3 = \{0\}$.

$p \in SingM$ is of the type $K_{\alpha,\beta,\gamma}$, $\alpha, \beta, \gamma = 0, 1, 2, 3$, if there exists $\phi \in AtlM$, $p \in D_\phi$, such that $\phi(p) \in (L_\alpha \cap L_\beta \cap L_\gamma) \backslash (L_0 \cap L_1 \cap L_2 \cap L_3)$.

$p \in SingM$ is of the type $K_{\alpha,\beta}$, $\alpha, \beta = 0, 1, 2, 3$, if there exists $\phi \in AtlM$, $p \in D_\phi$, such that $\phi(p) \in (L_\alpha \cap L_\beta) \backslash (L_\gamma \cap L_\delta)$, where $\alpha, \beta, \gamma, \delta$ are different indices from the set $\{0, 1, 2, 3\}$.

It is obvious that $RegM$ is open and dense in $M$, and $SingM$ is closed and boundary in $M$. $M$ can be represented as the disjoint sum

$$M = \bigcup_{i=0}^{2} M_i$$

where $M_0$ is the set of all singular points of the type $K_{0,1,2,3}$; $M_1$ is the set of all singular points of the type $K_{\alpha,\beta,\gamma}$; and $M_2$ is the set of all singular points of the type $K_{\alpha,\beta}$. Each of the spaces $M_i$ is of constant (differential) dimension $i$ (i.e., $dim\ M_i = dim\ T_pM_i = i$, $p \in M_i$).

We remember that $\eta_L$ is a Minkowski metric on the modeling space $L$. For any $p \in M$ we choose an $\mathcal{M}$-chart $\phi \in AtlM$ with the domain $D_\phi$. Of course, $g_\phi = \phi^*\eta_L$ is the Lorentz metric on $D_\phi$. We conclude that a Lorentz metric exists *locally* on the spacetime foam $(M, \tau)$. [A Lorentz metric *globally* on $(M, \tau)$ can be understood as an indexed family of Lorentz metrics defined locally (on the images of $\mathcal{M}$-charts) satisfying the correct transition rules; however, this concept will not be needed in the following.]

Let $\mathcal{X}(M)$ be the $\mathcal{C}(M)$-module of smooth vector fields tangent to the spacetime foam $(M, \tau, \mathcal{C})$, and let us consider the local behavior of tangent vector fields near singular points.

Let $\phi \in AtlM$ be an $\mathcal{M}$-chart with the domain $D_\phi$, $p \in D_\phi$ a singular point of the type $K_{\alpha,\beta}$, and $Y$ the representation of the tangent vector field $X \in \mathcal{X}(M)$ in the $\mathcal{M}$-chart $\phi$: $D_\phi \to V$. The vector field $Y$ is tangent to the subspaces $L_\alpha \cap V$ and $L_\beta \cap V$, and, of course, it is also tangent to the "edge" $L_\alpha \cap L_\beta \cap V$. Therefore, the tangent vector field $Y$ determines the pair of tangent vector fields $Y_\alpha = Y | L_\alpha \cap V$ and $Y_\beta = Y | L_\beta \cap V$, and these two vector fields are consistent, i.e., $Y_\alpha | L_\alpha \cap V = Y_\beta | L_\beta \cap V$. And conversely, a pair of consistent tangent vector fields $Y_\alpha$ and $Y_\beta$ determines exactly one tangent vector field $X | D_\phi$ such that its representation in the $\mathcal{M}$-chart $\phi$ is given by the tangent vector fields $Y_\alpha$ and $Y_\beta$. Similar analysis can be carried out if $p \in D_\phi$ is a singular point of the type $K_{\alpha,\beta,\gamma,\delta}$ or $K_{0,1,2,3}$. In such cases any tangent vector field $X \in \mathcal{X}(M)$ can locally be represented by three vector fields $(Y_\alpha, Y_\beta, Y_\gamma)$ or by four vector fields $(Y_0, Y_1, Y_2, Y_3)$, respectively.

Having the Lorentz metric and tangent vector fields, we can locally define the metric connection on the spacetime foam $(M, \tau, \mathcal{C})$. If $g_\phi$ is the Lorentz metric given by the $\mathcal{M}$-chart $\phi: D_\phi \to V$, there exists exactly one *connection* $\overset{\phi}{\nabla}: \mathcal{X}(D_\phi) \times \mathcal{X}(D_\phi) \to \mathcal{X}(D_\phi)$ such that

(i) $Zg_\phi(X, Y) = g_\phi(\overset{\phi}{\nabla}_Z X, Y) + g_\phi(X, \overset{\phi}{\nabla}_Z Y)$
(ii) $\overset{\phi}{\nabla}_X Y = \overset{\phi}{\nabla}_Y X + [X, Y]$
for any $X, Y, Z \in \mathcal{X}(D_\phi)$.

We will show this for the case when $p \in D_\phi$ is a singular point of the type $K_{\alpha,\beta,\gamma}$. Let $X, Y \in \mathcal{X}(D_\phi)$ be any tangent vector fields. We choose their representations $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ and $(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3)$ in the $\mathcal{M}$-chart $\phi$. The Lorentz metric $\eta_L$ induces the metrics $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ on $L_1 \cap D_\phi$, $L_2 \cap D_\phi$, $L_3 \cap D_\phi$. Let $\tilde{\nabla}_1, \tilde{\nabla}_2, \tilde{\nabla}_3$ be the Levi-Civita connections of the metrics $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$, respectively. It is easy to see that the vector fields

$$\tilde{\nabla}_{1\tilde{X}_1}\tilde{Y}_1, \quad \tilde{\nabla}_{2\tilde{X}_2}\tilde{Y}_2, \quad \tilde{\nabla}_{3\tilde{X}_3}\tilde{Y}_3$$

are consistent with each other. Consequently, there exists exactly one vector field $\overset{\phi}{\nabla}_X Y \in \mathcal{X}(D_\phi)$ having the representation

$$\tilde{\nabla}_{1\tilde{X}_1}\tilde{Y}_1, \quad \tilde{\nabla}_{2\tilde{X}_2}\tilde{Y}_2, \quad \tilde{\nabla}_{3\tilde{X}_3}\tilde{Y}_3$$

which ends the proof.

## 3. GENERAL COVARIANCE IN THE CLASSICAL SPACETIME FOAM

To disclose the meaning of the GC in the case of the spacetime foam $(M, \tau)$ we should determine the group *Iso* of this spacetime. For simplicity we shall assume that the entire curvature of the spacetime foam is concentrated in *SingM*, i.e., that the Riemann tensor vanishes on *RegM*.

Let $\mathbf{R}^4_1 = (\mathbf{R}^4, \eta)$ be a Minkowski space carrying the Lorentz metric $\eta$. The linear isometries of $\mathbf{R}^4_1$ form the Lorentz group $O_1(4)$, which is a subgroup of the isometry group $I(\mathbf{R}^4_1)$ of $\mathbf{R}^4_1$ (the Poincaré or inhomogeneous Lorentz group). Let us consider the isometry group $I(L^4_1)$ of the modeling space $L^4_1 = (L, \iota^*_L\eta)$. Since $L^4_1$ is a subspace of $\mathbf{R}^4_1$ its isometry group $I(L^4_1)$ is isomorphic with the subgroup $I_L(\mathbf{R}^4_1)$ of the group $I(\mathbf{R}^4_1)$ consisting of all isometries $a \in I(\mathbf{R}^4_1)$ which preserve $L^4_1$, i.e., such that $a(L^4_1) = L^4_1$. Of course, any such isometry transforms a singular point of a given type into the singular points of the same type. The singular point $p = (0, 0, 0, 0)$, as an isolated point, is the fixed point of the isometry $a$.

For any isometry $a: L_1^4 \to L_1^4$ there exists an extension $\bar{a}: \mathbf{R}_1^4 \to \mathbf{R}_1^4$ which for the point $p = (0, 0, 0, 0)$ is of the form $\bar{a} = T_0 \circ h$, where $h \in O_1(4)$ and $T_0 = id_{\mathbf{R}_1^4}$. We conclude that the isometry group $I(L_1^4)$ of the modeling space $L_1^4$ is isomorphic with the subgroup $O_1^L(4)$ of the Lorentz group $O_1(4)$, preserving $L_1^4$, modulo the subgroup $K_L^4 \subset O_1(4)$ of isometries which are identities on $L_1^4$, i.e., if $h \in K_L^4$, then $h \mid L_1^4 = id_L$. Therefore, we have

$$I(L_1^4) \simeq O_1(4)/K_L^4$$

As we can see, the group $I(L_1^4)$ excludes nontrivial translations. It is a discrete subgroup of $I(\mathbf{R}_1^4)$.

Now, we can describe the group $Iso(M, \tau)$ of the spacetime foam $(M, \tau)$. Of course, any isometry $\alpha: M \to M$ transforms singular points of a given type into the singular points of the same type, and locally determines the isometries of neighborhoods of singular points. Any point $q \in RegM$ has locally the isometry group isomorphic with the subgroup $I(\mathbf{R}_1^3)$ or $I(\mathbf{R}^3)$ of the Poincaré group $I(\mathbf{R}_1^4)$, depending on which sector of the "bubble" $q$ is situated. To consider the situation at a singular point $p \in SingM$, let $\phi \in AtlM$, $\phi: D_\phi \to V$, be an $\mathcal{M}$-chart such that $p \in D_\phi$. Then the isometry group locally at $p$ is of the form $O_1^V(4)/K_V^4$, where $O_1^V(4)$ is the isometry subgroup of $O_1(4)$ preserving $V$, and $K_V^4 \subset O_1(4)$ is the subgroup of isometries which are identities on $V$.

The group $Iso(M, \tau)$ is the symmetry group of the spacetime foam $(M, \tau)$, and it determines the meaning of GC in this spacetime. This is another way of saying that all $\mathcal{M}$-charts belonging to the (generalized) maximal atlas $AtlM$ on $(M, \tau)$ should be treated on an equal footing.

## 4. FREE FALL IN THE CLASSICAL SPACE-TIME FOAM

To study PE in the classical spacetime foam we need a theory of curves, geodesics in particular, in this space.

Let $(M, \tau)$ be a spacetime foam and $c: I \to M$ a curve in it. We say that a curve *does not change the sector* if its representation in an $\mathcal{M}$-chart $\phi \in AtlM$, $c_\phi = \phi \circ c \mid I_\phi$, where $I_\phi = c^{-1}(\text{Im } c \cap D_\phi)$, always remains in the subspace $L_\kappa$, $\kappa = 0, 1, 2, 3$, in the modeling space, and if this is valid for all $\mathcal{M}$-charts needed to "cover" $c(I)$. We shall also speak, in the obvious sense, about curves *changing the sectors*. Of course, this can happen only when a given curve passes through a singularity.

By a *geodesic* with respect to the Lorentz metric $g$ we understand a curve $c$ such that, for any $\mathcal{M}$-chart $\phi \in AtlM$, the representation $c_\phi$ of $c$ in the $\mathcal{M}$-chart $\phi$ is a geodesic with respect to the Lorentz metric $g_\phi$. In other words, a curve $c$ is a geodesic in $M$ if it is a geodesic in $M\backslash SingM$.

*Proposition 4.1.* If $c$: $(-\epsilon, \epsilon) \to M$ is a smooth curve such that $p = c(0) \in SingM$, $c(t) \in RegM$, for $t \neq 0$, and $c$ changes the sector at $c(0)$, then $c'(0)$ is a tangent vector to $SingM$.

[Of course, smoothness is understood here in the sense of the theory of structured spaces. A continuous mapping $f$: $M \to N$ of a structured space $(M, \mathscr{C})$ into a structured space $(N, \mathscr{D})$ is said to be *smooth* if, for any cross section $g \in \mathscr{D}(U)$, $U \in topN$, one has $g \circ (f|f^{-1}(U)) \in \mathscr{C}(f^{-1}(U))$; see Heller and Sasin (1995).]

*Proof.* Let us suppose that $c_\phi$ changes the sector $L_1$ into the sector $L_2$. We divide $c_\phi$ into two parts, $c_1$: $(-\epsilon, 0] \to L_1$ and $c_2$: $[(0, \epsilon, ) \to L_2$. We have $c_1'(0) \in T_pL_1$, $c_2'(0) \in T_pL_2$, and, from the smoothness, $c'(0) = c_1'(0) = c_2'(0) \in T_pL_1 \cap T_pL_2 \cong T_p(L_1 \cap L_2)$. ■

This result is a close analogue of the one obtained by Vickers (1990), who demonstrated that, in the case of quasiregular singularities of generalized cosmic strings, the tangent space is degenerate only in normal directions to the singularity. It can be easily seen that if a smooth curve changes the "edges," for instance, from $L_0 \cap L_1$ into $L_2 \cap L_3$, passing through the singular point $\{0, 0, 0, 0\}$, then $c'(0) = 0$ (proof is similar to that of Proposition 4.1).

Taking into account the above-discussed behavior of curves, we are entitled to say that geodesics which do not change the sector and geodesics which enter singularities tangentially should be regarded as the histories of freely falling test particles. In this case, there is nothing new as compared with the standard theory of general relativity.

In agreement with Proposition 4.1, a curve $c$ can also (smoothly!) change the sector by slowing down to zero at the singularity, i.e., $c'(p) = 0$, $p \in SingM$. In such a case either the test particle, whose history is the curve $c$, will always remain at $p$, or an extra force must be applied to initiate the further motion. We see, therefore, that such a curve is not a geodesic and cannot represent a free fall. A new force, besides gravity, is needed to slow down on approaching the singularity and to accelerate to move from it.

If a test particle freely falling toward a singularity from a nontangential direction does not slow down to zero, it will experience a "shock" at the singularity, i.e., its history will cease to be smooth. Formally, it is a *smooth* geodesic (i.e., smooth in $M\backslash SingM$) and from the physical point of view it should be qualified as freely falling since no force, besides gravity, is acting. However, in this case, the gravitational field cannot be eliminated by a suitable choice of a reference frame. The usual version of EP (in both its weak and strong formulations) is evidently violated.

It is commonly accepted that the geometric counterpart of the fact that a gravitational field can locally be transformed away is the existence of the flat Minkowski tangent space at the point in question (point $p$, say) and at all sufficiently nearby points of the considered spacetime. Our toy model of the spacetime foam shows that it is not enough. Another, so far implicitly assumed, condition is that the dimension of all these flat Minkowski tangent spacetimes (in a close neighborhood of the point $p$) should be the same. This condition is clearly not satisfied in the case of the classical spacetime foam (the dimension of tangent spaces at points of $SingM$ is not constant and evidently different than that of tangent spaces at points of $RegM$). This is a geometric counterpart of the fact that the gravitational field can only be transformed away in tangent directions to singularities.

As we have seen in Section 2, one can define a Lorentz metric on a spacetime foam $(M, \tau)$ since its modeling space $L$, and consequently the spacetime foam $(M, \tau)$ itself, can locally be embedded in the Minkowski spacetime equipped with the Minkowski metric $\eta$. Heller (1993) proved, more generally, that if $(M, \tau, \mathscr{C})$ is a sheaf space and if a (pseudo)Riemannian metric exists locally at $p \in U \subset \tau$, then $dimT_pM < \infty$ for any $p \in U$, and $(M, \tau, \mathscr{C})$ can locally be embedded in a flat (pseudo)Euclidean space (in fact, this result has been proved for differential spaces, but the generalization to sheaf spaces is immediate). To improve our model we can consider a spacetime foam $(M, \tau)$ which can be locally embedded in a (curved) spacetime manifold $(\tilde{M}, \tilde{g})$. Since at each point $p$ of $\tilde{M}$ there exists the tangent flat Minkowski spacetime, $(M, \tau)$ can also be locally embedded in this flat Minkowski spacetime.

To make our toy model more plausible (but still a toy model), let us consider many classical spacetime foams $\{(M, \tau_i, \mathscr{C}_j)\}$, $i, j = 1, 2, \ldots$, with the restriction that all of them can locally be embedded in the same spacetime manifold $(\tilde{M}, \tilde{g})$. [The "foam of topologies" $(M, \tau_i)$ was studied by Isham (1989); here we propose that also the "differential structure" $\mathscr{C}$ can be foamy; for the time being we neglect the question of how the "foam of topologies" and the "foam of differential structures" should be synchronized (they are not independent)]. We could regard all these spacetime foams $\{(M, \tau_i, \mathscr{C}_j)\}$ as a fluctuating quantized spacetime on the microscopic level (say, at the level of the Planck length and Planck time), and the spacetime manifold $(\tilde{M}, \tilde{g})$ as a suitable macroscopic "averaging" of these fluctuations. To substantiate our claim we should elaborate the averaging procedure. However, we postpone doing that until a more realistic model is available.

This picture allows us to generalize the usual formulation of EP: *the microscopic spacetime must be such that it could be possible to (locally) embed it in the spacetime manifold* $(\tilde{M}, \tilde{g})$. If this is true, the distinction between weak and strong formulations of EP makes sense only on the macroscopic level.

One can construct many examples of sheaf spaces which can be embedded in no finite-dimensional manifold (Sasin, 1988). Such spaces are forbidden by the generalized EP. The reason is that they could not be suitably averaged to give a macroscopic spacetime manifold.

A word of warning must be added. Since our model is only a toy model, we do not attach great value to the above-formulated generalized version of EP. What seems to us important is that even this naive model clearly demonstrates that when changing to the quantum level, EP should be generalized rather than kept in its traditional form.

## 5. EINSTEIN'S EQUATIONS ON THE CLASSICAL SPACETIME FOAM

Tensor fields and differential forms of all types can be defined in the classical spacetime foam $(M, \tau)$, in principle, in the same manner as the Lorentz metric tensor was defined in it in Section 2 [tensor fields and differential forms on sheaf spaces in general were studied in Heller and Sasin (1995)]. An important fact is that if they are defined on *RegM*, they have unique prolongations to $M$. As an example, we shall prove this for tensor fields of the type $(n, 0)$. Let $\omega_1: \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{C}(M)$ and $\omega_2: \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{C}(M)$ be two such vector fields. We shall show that if $\omega_1 | RegM = \omega_2 | RegM$, then $\omega_1 = \omega_2$. Indeed, if $X_1, \ldots, X_n \in \mathcal{X}(M)$, one has

$$\begin{aligned}\omega_1(X_1, \ldots, X_n) | RegM &= (\omega_1 | RegM)(X_1 | RegM, \ldots, X_n | RegM) \\ &= (\omega_2 | RegM)(X_1 | RegM, \ldots, X_n | RegM) = \omega_2(X_1, \ldots, X_n) | RegM)\end{aligned}$$

From the continuity and the fact that *RegM* is open and dense in $M$ it follows that $\omega_1(X_1 \ldots, X_n) = \omega_2(X_1, \ldots, X_n)$. Similar proofs can be repeated for any tensor fields and differential forms.

Let us consider a tensorial equation $w(T_1, \ldots, T_n) = 0$ with the polynomial dependence between the tensors $T_1, \ldots, T_n$ defined on *RegM*. We have:

*Proposition 5.1.* If the tensor fields $T_1, \ldots, T_n$ can be uniquely prolonged to $M = RegM \cup SingM$, then the equation $w(T_1, \ldots, T_n) = 0$ is valid on $M$.

*Proof.* Let $\overline{T}_1, \ldots, \overline{T}_n$ be prolongations of $T_1, \ldots, T_n$ to $M$. Let us notice that

$$\begin{aligned}w(\overline{T}_1, \ldots, \overline{T}_n) | RegM &= (w | RegM)(\overline{T}_1 | RegM, \ldots, \overline{T}_n | RegM) \\ &= (w | RegM)(T_1, \ldots, T_n) = 0\end{aligned}$$

Therefore, $w(\overline{T}_1, \ldots, \overline{T}_n) = 0$. ■

An important corollary of the above proposition is that if ($M$, $\tau$) is the classical spacetime foam and Einstein's equations are defined on *RegM*, they can be defined on $M$. We have, therefore, a true generalization of Einstein's field equations on a spacetime with singularities (see also Heller and Sasin 1995; Heller, 1992), and we can consider only those spacetime foams which are solutions to these generalized Einstein equations.

Let $\mathscr{E}$ be the set of all solutions of Einstein's equations defined on the spacetime foam ($M$, $\tau$), and *IsoM* the group of all isomorphisms of ($M$, $\tau$) as discussed in Section 3. In agreement with GC, we must ascribe physical meaning to $\mathscr{E}/IsoM$ rather than to $\mathscr{E}$ itself. Of course, after "averaging" to the macroscopic spacetime (modeled by the usual manifold) one recovers the ordinary principle of GC.

## 6. GRAVITY AT SHORT RANGE

Our classical spacetime foam model exhibits an interesting property. Singularities inherent in the structure of the spacetime foam can be regarded as sources of a gravitational field, but this field corresponds to a short-range force, and does this in a very peculiar manner. Let ($M$, $\tau$) be a spacetime foam (possibly satisfying the generalized Einstein equations). For the time being let us assume that the curvature tensor on *RegM* vanishes, and the entire curvature is concentrated in singularities. Let us imagine that a test particle travels in *RegM* toward a singularity (along a nontangential direction). The particle has no warning that it approaches the singularity until it hits it. The gravitational field generated by the singularity is a short-range field in the sense that no particle (or photon) feels this field before it arrives at the singularity. This situation is typical for quasiregular singularities (Ellis and Schmidt, 1977), but in the present context of our naive spacetime model it acquires a new meaning. One could speculate that at the singularities of the spacetime foam gravitons become massive particles. (A similar analysis could be carried out with regard to photons. Since, however, our model is far from being realistic, we shall not go into details.)

If the curvature on *RegM* does not vanish, the particle approaching the singularity feels only the usual "long-range" gravity, but it still does not feel the "short-range" (massive?) gravity "located" at the singularity until it reaches it [the situation is similar to that in a neighborhood of the singularity due to the "generalized cosmic string"; see Vickers (1987)]. In the realm of macroscopic physics, gravity reveals only its long-range aspect. In agreement with the main idea of general relativity, gravity is connected with the spacetime curvature. That part of the curvature which is of the singular type (which is concentrated at the edges and vertices of the foam) has a short-range character;

that part of the curvature which is of the usual tensorial character gives rise to the standard long-range gravitational force.

In connection with the above, it is interesting to notice that it was Weinberg (1965) who stipulated that at the quantum level EP (in its usual formulation) is equivalent to the fact that gravity is massless and has a Coulomb-type potential. If this is true, any modification of EP could modify the long-range character of gravity. Weinberg based his arguments on the perturbation theory of the Lorentz-invariant $S$-matrix, whereas our picture is rooted in purely geometrical considerations.

## REFERENCES

Anderson, J. L. (1967). *Principles of Relativity Physics,* Academic Press, New York.

Berezin, F. A. (1983). *Introduction to the Algebra and Analysis with Anticommuting Variables,* Moscow University Press, Moscow [in Russian].

Candelas, P., and Sciama, D. W. (1984). Is there a quantum equivalence principle? in *Quantum Theory of Gravity—Essays in Honor of the 60th Birthday of Bryce S DeWitt,* S. M. Christensen, ed., Hilger, Bristol.

Choquet-Bruhat, Y. (1989). *Graded Algebras and Supermanifolds,* Bibliopolis, Naples.

DeWitt, B. (1984). *Supermanifolds,* Cambridge University Press, Cambridge.

Ellis, G. F. R., and Schmidt, B. (1977). *General Relativity and Gravitation,* **11**, 915.

Hartle, J. B., and Hawking, S. W. (1983). *Physical Review D,* **28**, 2960.

Hawking, S. W. (1982). The boundary conditions of the universe, in *Astrophysical Cosmology,* H. A. Brück, G. V. Coyne, and M. S. Longair, eds., Pontifical Academy, Scripta Varia, Vatican City.

Heller, M. (1992). *International Journal of Theoretical Physics,* **31**, 277.

Heller, M. (1993). *Acta Physica Polonica,* **24B**, 911.

Heller, M., and Sasin, W. (1994). *General Relativity and Gravitation,* **26**, 797.

Heller, M., and Sasin, W. (1995). *Journal of Mathematical Physics,* **36**, 3644.

Isham, C. J. (1989). *Classical and Quantum Gravity,* **6**, 1509.

Raine, D. J. (1981). *The Isotropic Universe,* Hilger, Bristol.

Raine, D. J., and Heller, M. (1981). *The Science of Space-Time,* Pachart, Tucson.

Rovelli, C. (1991a). *Classical and Quantum Gravity,* **8**, 297.

Rovelli, C. (1991b). *Classical and Quantum Gravity,* **8**, 317.

Rovelli, C. (1991c). *Classical and Quantum Gravity,* **8**, 1613.

Sasin, W. (1988). *Demonstratio Mathematica* (Warsaw), **21**, 897.

Taylor, J. G. (1979). *Physical Review D,* **19**, 2336.

Torretti, R. (1983). *Relativity and Geometry,* Pergamon Press, Oxford.

Vickers, J. A. G. (1987). *Classical and Quantum Gravity,* **4**, 1.

Vickers, J. A. G. (1990). *Classical and Quantum Gravity,* **7**, 731.

Weinberg, S. (1965). *Physical Review,* **138B**, 988.

Weinberg, S. (1972). *Gravitation and Cosmology,* Wiley, New York.